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THERMAL STRESSES IN DOUBLY-CURVED CROSS-PLY LAMINATES

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Abstract—The elastic response of a doubly-curved cross-ply laminated panel subject to mechanical loading and temperature variation is investigated. The three-dimensional equilibrium equations, expressed in terms of displacements, are reduced to a system of coupled ordinary differential equations, which are then solved using the power series method. Numerical results are presented for a traction loaded saddle-shape shell and for a heated spherical panel.

INTRODUCTION

When exact solutions to three-dimensional elasticity theory are available, they are valuable not only in their own right, but also as useful benchmarks for verifying mathematical procedures leading to approximate solutions, and for assessing the validity of approximate formulations (e.g. beam, plate or shell theories). In the case of bidirectional composites and sandwich plates, Pagano (1970) developed an exact solution, and compared it with a solution based on classical laminated plate theory. Srinivas and Rao (1970) established an exact analysis for bending, vibration and buckling of flat laminates. More recently, an exact three-dimensional thermoelasticity solution for a cross-ply cylindrical panel was obtained by Huang and Tauchert (1991) using the power series method. A similar procedure is employed here to derive the thermoelasticity solution for a doubly-curved cross-ply laminate.

The laminate under consideration consists of N layers of a unidirectionally reinforced material (Fig. 1) with each layer taken to be macroscopically homogeneous and orthotropic. The radii of curvature of the middle surface in the θ_2- and θ_3-directions, denoted by R_2 and

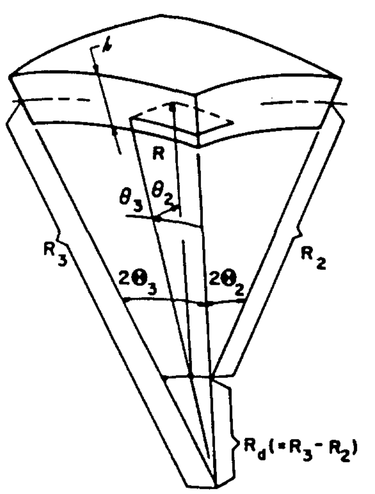

Fig. 1. Doubly-curved laminated panel.

R_3 $(= R_2+R_d)$, are taken to be constant. The coordinate R represents the radius of curvature in the θ_2-direction of an arbitrary point in the shell; thus $R = R_2$ if the point lies on the middle surface. The middle surface of the doubly-curved panel is assumed to have a Euclidean geometry, i.e. $(ds)^2 = (R_2\,d\theta_2)^2+(R_3\,d\theta_3)^2$. It is important to mention that this metric equation leads to an approximate formulation in the case of shallow doubly-curved surfaces. Hereafter, coordinates (R, θ_2, θ_3) will be also referred to as i-coordinates $(i = 1, 2, 3)$.

GOVERNING EQUATIONS

In the following analysis, the non-dimensional quantities

$$r = R/R_2, \quad r_d = R_d/R_2, \quad u_i = v_i/R_2 \tag{1}$$

are employed. Here v_i is the component of displacement in the i-coordinate direction.

The stress–strain–temperature relations for each orthotropic lamina, written in contracted notation $(\sigma_1 = \sigma_{11}, \sigma_2 = \sigma_{22}, \ldots, \sigma_4 = \sigma_{23}$, etc.) are

$$\sigma_i = C_{ij}(\varepsilon_j-\alpha_j T), \quad (i, j = 1\text{–}6), \tag{2}$$

where C_{ij} denote the elastic moduli, α_j are the coefficients of thermal expansion, T is the temperature rise from the stress-free state; the repeated index implies summation.

The equilibrium equations are (Huang, 1990):

$$\begin{aligned}
&\sigma_{11,1}+\frac{1}{r}\sigma_{12,2}+\frac{1}{r+r_d}\sigma_{13,3}+\frac{\sigma_{11}-\sigma_{22}}{r}+\frac{\sigma_{11}-\sigma_{33}}{r+r_d} = 0\\
&\sigma_{12,1}+\frac{2}{r}\sigma_{12}+\frac{1}{r+r_d}\sigma_{12}+\frac{1}{r}\sigma_{22,2}+\frac{1}{r+r_d}\sigma_{23,3} = 0\\
&\sigma_{13,1}+\frac{1}{r}\sigma_{13}+\frac{2}{r+r_d}\sigma_{13}+\frac{1}{r}\sigma_{23,2}+\frac{1}{r+r_d}\sigma_{33,3} = 0
\end{aligned} \tag{3}$$

in which a comma denotes differentiation.

The strain–displacement relationships are

$$\begin{aligned}
&\varepsilon_{11} = u_{1,1}, \quad \varepsilon_{22} = \frac{1}{r}(u_{2,2}+u_1), \quad \varepsilon_{33} = \frac{1}{r+r_d}(u_{3,3}+u_1)\\
&\gamma_{12} = u_{2,1}+\frac{1}{r}u_{1,2}-\frac{1}{r}u_2, \quad \gamma_{13} = u_{3,1}+\frac{1}{r+r_d}u_{1,3}-\frac{1}{r+r_d}u_3, \quad \gamma_{23} = \frac{1}{r+r_d}u_{2,3}+\frac{1}{r}u_{3,2}.
\end{aligned} \tag{4}$$

By substituting eqns (2) and (4) into eqn (3), the equilibrium equations become

$$\begin{aligned}
C_{11}u_{1,11}&+\frac{C_{66}}{r^2}u_{1,22}+\frac{C_{55}}{(r+r_d)^2}u_{1,33}+C_{11}\left(\frac{1}{r}+\frac{1}{r+r_d}\right)u_{1,1}\\
&+\left[-\frac{C_{22}}{r^2}-\frac{C_{33}}{(r+r_d)^2}+\frac{1}{r(r+r_d)}(C_{12}+C_{13}-2C_{23})\right]u_1\\
&+(C_{12}+C_{66})\frac{1}{r}u_{2,12}+\left(\frac{C_{12}-C_{23}}{r+r_d}-\frac{C_{22}+C_{66}}{r}\right)\frac{1}{r}u_{2,2}\\
&+\frac{C_{13}+C_{55}}{r+r_d}u_{3,13}+\left(\frac{C_{13}-C_{23}}{r}-\frac{C_{33}+C_{55}}{r+r_d}\right)\frac{1}{r+r_d}u_{3,3} = f(r, \theta_2, \theta_3)
\end{aligned}$$

$$(C_{12}+C_{66})\frac{1}{r}u_{1,12}+\left(\frac{C_{23}+C_{66}}{r+r_d}+\frac{C_{22}+C_{66}}{r}\right)\frac{1}{r}u_{1,2}$$
$$+C_{66}u_{2,11}+\frac{C_{22}}{r^2}u_{2,22}+\frac{C_{44}}{(r+r_d)^2}u_{2,33}+C_{66}\left(\frac{1}{r}+\frac{1}{r+r_d}\right)u_{2,1}$$
$$-C_{66}\left(\frac{1}{r}+\frac{1}{r+r_d}\right)\frac{1}{r}u_2+\frac{C_{23}+C_{44}}{r(r+r_d)}u_{3,23}=g(r,\theta_2,\theta_3)$$

$$\frac{C_{13}+C_{55}}{r+r_d}u_{1,13}+\left(\frac{C_{23}+C_{55}}{r}+\frac{C_{33}+C_{55}}{r+r_d}\right)\frac{1}{r+r_d}u_{1,3}$$
$$+\frac{C_{23}+C_{44}}{r(r+r_d)}u_{2,23}+C_{55}u_{3,11}+\frac{C_{44}}{r^2}u_{3,22}+\frac{C_{33}}{(r+r_d)^2}u_{3,33}$$
$$+C_{55}\left(\frac{1}{r}+\frac{1}{r+r_d}\right)u_{3,1}-C_{55}\left(\frac{1}{r}+\frac{1}{r+r_d}\right)\frac{1}{r}u_3=h(r,\theta_2,\theta_3) \quad (5)$$

where

$$f(r,\theta_2,\theta_3)=\sigma^T_{1,1}+\frac{1}{r}(\sigma^T_1-\sigma^T_2)+\frac{1}{r+r_d}(\sigma^T_1-\sigma^T_3)$$
$$g(r,\theta_2,\theta_3)=\frac{1}{r}\sigma^T_{2,2},\quad h(r,\theta_2,\theta_3)=\frac{1}{r+r_d}\sigma^T_{3,3} \quad (6)$$

in which the quantities σ^T_i are defined by

$$\sigma^T_i=C_{ij}\alpha_jT,\quad (i,j=1,2,3). \quad (7)$$

THERMOELASTIC ANALYSIS

The laminate is taken to be simply supported in such a manner that all edges are fixed against tangential displacements but remain free to translate in the normal (in-plane) direction, in which case:

$$\text{at}\quad \theta_2=0,2\Theta_2:\quad u_1=u_3=\sigma_{22}=0;$$
$$\text{at}\quad \theta_3=0,2\Theta_3:\quad u_1=u_2=\sigma_{33}=0. \quad (8)$$

Since most loadings can be represented in Fourier series form, the boundary conditions for the inner ($r=r_i$) and outer ($r=r_o$) lateral surfaces are expressed as:

$$\text{at}\quad r=r_i:\quad \sigma_{11}=q_{1i}\sin\alpha\theta_2\sin\beta\theta_3,\quad \sigma_{12}=q_{2i}\cos\alpha\theta_2\sin\beta\theta_3$$
$$\sigma_{13}=q_{3i}\sin\alpha\theta_2\cos\beta\theta_3;$$
$$\text{at}\quad r=r_o:\quad \sigma_{11}=q_{1o}\sin\alpha\theta_2\sin\beta\theta_3,\quad \sigma_{12}=q_{2o}\cos\alpha\theta_2\sin\beta\theta_3$$
$$\sigma_{13}=q_{3o}\sin\alpha\theta_2\cos\beta\theta_3 \quad (9)$$

where $\alpha=m\pi/(2\Theta_2)$ and $\beta=n\pi/(2\Theta_3)$, with m and n considered arbitrary. Similarly, the temperature variation is described by

$$T = T_1(r)\sin\alpha\theta_2 \sin\beta\theta_3. \tag{10}$$

The displacements are taken in the form:

$$\begin{aligned} u_1(r,\theta_2,\theta_3) &= U_1(r)\sin\alpha\theta_2\sin\beta\theta_3,\\ u_2(r,\theta_2,\theta_3) &= U_2(r)\cos\alpha\theta_2\sin\beta\theta_3,\\ u_3(r,\theta_2,\theta_3) &= U_3(r)\sin\alpha\theta_2\cos\beta\theta_3. \end{aligned}\tag{11}$$

The assumed displacement field (11) satisfies all of the edge conditions (8).

By substituting eqns (11) into eqns (5), the partial differential equations are reduced to the following ordinary differential equations in the variable r:

$$\begin{aligned} &C_{11}r(rU_1')'+[C_{11}\bar{r}]rU_1'+\left[-C_{22}-\alpha^2C_{66}+\left(\frac{C_{12}}{2}+C_{13}-2C_{23}\right)\bar{r}-(C_{33}+\beta^2C_{55})(\bar{r})^2\right]U_1\\ &\quad-[\alpha(C_{12}+C_{66})]rU_2'-\alpha[-C_{22}-C_{66}+(C_{12}-C_{23})\bar{r}]U_2-[\beta(C_{13}+C_{55})\bar{r}]rU_3'\\ &\qquad-\beta[(C_{13}-C_{23})\bar{r}-(C_{33}+C_{55})(\bar{r})^2]U_3 = F(r)\\ &[\alpha(C_{12}+C_{66})]rU_1'+\alpha[C_{22}+C_{66}+(C_{23}+C_{66})\bar{r}]U_1\\ &\quad+C_{66}r(rU_2')'+[C_{66}\bar{r}]rU_2'-[C_{66}+\alpha^2C_{22}+C_{66}\bar{r}+\beta^2C_{44}(\bar{r})^2]U_2\\ &\qquad-[\alpha\beta(C_{23}+C_{44})\bar{r}]U_3 = G(r)\\ &[\beta(C_{13}+C_{55})\bar{r}]rU_1'+\beta[(C_{23}+C_{55})\bar{r}+(C_{33}+C_{55})(\bar{r})^2]U_1\\ &\quad-[\alpha\beta(C_{23}+C_{44})\bar{r}]U_2+C_{55}r(rU_3')'+[C_{55}\bar{r}]rU_3'\\ &\qquad-[\alpha^2C_{44}+C_{55}\bar{r}+(C_{55}+\beta^2C_{33})(\bar{r})^2]U_3 = H(r) \end{aligned}\tag{12}$$

in which $(\ )' \equiv \mathrm{d}(\ )/\mathrm{d}r$, and

$$\begin{aligned} \bar{r} &= \frac{r}{r+r_d}\\ F(r) &= r^2T_{1,1}\Sigma_1+rT_1(\Sigma_1-\Sigma_2)+rT_1\bar{r}(\Sigma_1-\Sigma_3)\\ G(r) &= \alpha rT_1\Sigma_2,\quad H(r) = \beta\bar{r}rT_1\Sigma_3\\ \Sigma_i &= C_{ij}\alpha_j,\quad (i,j = 1,2,3). \end{aligned}\tag{13}$$

The value of r_d, the difference of the radii of curvature in the θ_2- and θ_3-directions, will dictate the form of solution of the differential eqns (12).

Case (1): $|r/r_d| < 1$.

By letting

$$W_i(r) = U_i(r),\quad W_{i+3}(r) = rU_i'(r) = rW_i'(r),\quad (i = 1,2,3) \tag{14}$$

eqns (12) can be written in the form:

$$rW_i'(r) = A_{ij}(r)W_j(r)+B_i(r),\quad (i,j = 1\text{-}6). \tag{15}$$

If all the coefficients $A_{ij}(r)$ are analytic at $r = 0$, then there exists a set of homogeneous solutions of the above equations, namely (Ince, 1956; Hartman, 1964):

$$W_i(r) = r^sV_i(r),\quad (i = 1\text{–}6), \tag{16}$$

where s is a certain constant, and $V_i(r)$ is analytic at the origin.

Note that the expressions enclosed in the brackets of eqns (12) are analytic at $r = 0$, as is each function $A_{ij}(r)$ in eqns (15). Therefore, at least one homogeneous solution of eqns (12) can be represented in the general form:

$$[U_1(r), U_2(r), U_3(r)] = r^p \sum_{m=0,1,2}^{\infty} [a_m, b_m, c_m] r^m. \tag{17}$$

Substituting (17) into the homogeneous form of (12), and equating the coefficients of the smallest power in each of the resulting equations, leads to the following system of equations:

$$\begin{aligned}(C_{11}p^2 - \alpha^2 C_{66} - C_{22})a_0 + \alpha[C_{22} + C_{66} - p(C_{12} + C_{66})]b_0 &= 0\\ \alpha[C_{22} + C_{66} + p(C_{12} + C_{66})]a_0 + (C_{66}p^2 - \alpha^2 C_{22} - C_{66})b_0 &= 0\\ [p^2 C_{55} - \alpha^2 C_{44}]c_0 &= 0.\end{aligned} \tag{18}$$

Nonzero values of the coefficients a_0, b_0 and c_0 can exist only if the determinant of eqns (18) is zero. The resulting equation is the indicial equation of the differential equations (12), namely

$$(p^2C_{55} - \alpha^2 C_{44}) \cdot (C_{11}C_{66}p^4 + p^2[\alpha^2(C_{12} + C_{66})^2 - C_{11}(\alpha^2 C_{22} + C_{66}) - C_{66}(\alpha^2 C_{66} + C_{22})] + C_{22}C_{66}(\alpha^2 - 1)^2) = 0. \tag{19}$$

Six roots (counting multiplicity) are expected from eqn (19). Upon substituting the indicial roots p_i back into eqns (18), two arbitrary coefficients can be expressed in terms of the third, provided that one is nonzero.

To avoid mathematical difficulties, only those cases in which real roots of the indicial equations are distinct and do not differ from each other by an integer are considered. The successive coefficients a_m, b_m and c_m ($m \geq 1$) in eqn (17) can be determined from the recursion equations [see Huang and Tauchert (1991)].

The general homogeneous solution of eqns (12), denoted by U_1^h, U_2^h and U_3^h, can be written as

$$[U_1^h(r), U_2^h(r), U_3^h(r)] = \sum_{i=1}^{6} e_i r^{p_i} \sum_{m=0,1}^{\infty} [a_{im}, b_{im}, c_{im}] r^m \tag{20}$$

where p_i are the roots of the indicial eqn (19). The unknown coefficients e_i are determined from the traction conditions prescribed on the shell's lateral surfaces and the continuity requirements for adjacent laminae.

Next, consider the particular solution of eqns (12). It is assumed that the right-hand sides of eqns (12) can be expressed in the polynomial form:

$$[F(r), G(r), H(r)] = r^s \sum_{n=0,1,2}^{\infty} [f_n, g_n, h_n] r^n. \tag{21}$$

Particular solutions U_1^p, U_2^p and U_3^p of eqns (12) are (Hildebrand, 1976):

$$[U_1^p(r), U_2^p(r), U_3^p(r)] = r^s \sum_{n=0,1,2}^{\infty} [a_n, b_n, c_n] r^n \tag{22}$$

provided that p_i is not equal to $(s+n)$.

Equating the coefficients of power r^{s+n} of eqns (12) to zero yields a set of equations for a_n, b_n and c_n, from which the particular solutions can be obtained.

For an N-layer laminate, there are a total of $6N$ undetermined coefficients. These can be determined by satisfying six surface traction conditions (involving σ_{11}, σ_{12}, σ_{13}), and satisfying $6(N-1)$ interface continuity requirements for σ_{11}, σ_{12}, σ_{13}, u_1, u_2 and u_3.

Case (2): $|r/r_d| > 1$.

Substitution of $\rho = 1/r$ and $\tilde{\rho} = 1/(1+r_d\rho)$ into eqns (12) gives for this case:

$$
\begin{aligned}
& C_{11}\rho(\rho U_1')' - [C_{11}\tilde{\rho}]\rho U_1' + [-C_{22} - \alpha^2 C_{66} + (C_{12}+C_{13}-2C_{23})\tilde{\rho} - (C_{33}+\beta^2 C_{55})(\tilde{\rho})^2]U_1 \\
& \quad + [\alpha(C_{12}+C_{66})]\rho U_2' - \alpha[-C_{22} - C_{66} + (C_{12}-C_{23})\tilde{\rho}]U_2 + [\beta(C_{13}+C_{55})\tilde{\rho}]\rho U_3' \\
& \qquad - \beta[(C_{13}-C_{23})\tilde{\rho} - (C_{33}+C_{55})(\tilde{\rho})^2]U_3 = F(\rho) \\
& -[\alpha(C_{12}+C_{66})]\rho U_1' + \alpha[C_{22}+C_{66}+(C_{23}+C_{66})\tilde{\rho}]U_1 + C_{66}\rho(\rho U_2')' - [C_{66}\tilde{\rho}]\rho U_2' \\
& \qquad - [C_{66} + \alpha^2 C_{22} + C_{66}\tilde{\rho} + \beta^2 C_{44}(\tilde{\rho})^2]U_2 - [\alpha\beta(C_{23}+C_{44})\tilde{\rho}]U_3 = G(\rho) \\
& -[\beta(C_{13}+C_{55})\tilde{\rho}]\rho U_1' - \beta[(C_{23}+C_{55})\tilde{\rho} + (C_{33}+C_{55})(\tilde{\rho})^2]U_1 - [\alpha\beta(C_{23}+C_{44})\tilde{\rho}]U_2 \\
& \quad + C_{55}\rho(\rho U_3')' - [C_{55}\tilde{\rho}]\rho U_3' - [\alpha^2 C_{44} + C_{55}\tilde{\rho} + (C_{55}+\beta^2 C_{33})(\tilde{\rho})^2]U_3 = H(\rho). \qquad (23)
\end{aligned}
$$

in which,

$$
\begin{aligned}
F(\rho) &= -T_{1,1}\Sigma_1 + \frac{1}{\rho}T_1(\Sigma_1 - \Sigma_2) + \frac{1}{\rho}T_1\tilde{\rho}(\Sigma_1 - \Sigma_3) \\
G(\rho) &= \alpha\frac{1}{\rho}T_1\Sigma_2, \quad H(\rho) = \beta\tilde{\rho}\frac{1}{\rho}T_1\Sigma_3. \qquad (24)
\end{aligned}
$$

The coefficients within brackets in eqns (23) are analytic at $\rho = 0$, and the corresponding power series expressions are convergent for $|r_d\rho| < 1$ (i.e. $|r/r_d| > 1$).

The homogeneous solutions of (23) are of the form

$$
[U_1(\rho), U_2(\rho), U_3(\rho)] = \rho^p \sum_{m=0,1,2}^{\infty} [a_m, b_m, c_m]\rho^m. \qquad (25)
$$

Consequently, the homogeneous solutions of eqns (12) can be expressed as

$$
[U_1(r), U_2(r), U_3(r)] = r^p \sum_{m=0,1,2}^{\infty} [a_m, b_m, c_m]r^{-m}. \qquad (26)
$$

Substituting eqns (26) into eqns (12), and setting the coefficients of the highest power (r^p) equal to zero, yields the following equations:

$$
\begin{aligned}
& [C_{11}p(p+1) - \alpha^2 C_{66} - \beta^2 C_{55} - C_{12} + C_{13} - C_{22} - 2C_{23} - C_{33}]a_0 \\
& \qquad + \alpha[-p(C_{12}+C_{66}) - C_{12} + C_{22} + C_{23} + C_{66}]b_0 \\
& \qquad\qquad + \beta[-p(C_{13}+C_{55}) - C_{13} + C_{23} + C_{33} + C_{55}]c_0 = 0 \\
& \alpha[p(C_{12}+C_{66}) + C_{22} + C_{23} + 2C_{66}]a_0 \\
& \qquad + [C_{66}p(p+1) - \alpha^2 C_{22} - \beta^2 C_{44} - 2C_{66}]b_0 - \alpha\beta(C_{23}+C_{44})c_0 = 0 \\
& \beta[p(C_{13}+C_{55}) + C_{23} + C_{33} + 2C_{55}]a_0 \\
& \qquad - \alpha\beta(C_{23}+C_{44})b_0 + [C_{55}p(p+1) - \alpha^2 C_{44} - \beta^2 C_{33}]c_0 = 0. \qquad (27)
\end{aligned}
$$

The nonvanishing coefficients a_0, b_0 and c_0 can exist only if the determinant of eqns (27) is zero. A closed form solution to the sixth-order determinant equation is available (see Appendix). The successive coefficients a_m, b_m and c_m ($m \geqslant 1$) can be obtained by equating the coefficients of power (r^{p-m}) to zero. The resulting general homogeneous solutions can be written as:

$$[U_1^h(r), U_2^h(r), U_3^h(r)] = \sum_{i=1}^{6} e_i r^{p_i} \sum_{m=0,1}^{\infty} [a_{im}, b_{im}, c_{im}] r^{-m}. \tag{28}$$

It is presumed that the right-hand sides of eqns (12) can be expressed by the following polynomials:

$$[F(r), G(r), H(r)] = r^q \sum_{n=0,1,2}^{\infty} [f_n, g_n, h_n] r^{-n}. \tag{29}$$

The particular solution takes the form

$$[U_1^p(r), U_2^p(r), U_3^p(r)] = r^q \sum_{n=0,1,2}^{\infty} [a_n, b_n, c_n] r^{-n} \tag{30}$$

if roots p_i of the indicial equation are not equal to $(q-n)$. Coefficients a_n, b_n and c_n can be determined by the procedure described earlier.

In the case of a spherical panel (a special class of doubly-curved shells) r_d is equal to zero, and eqns (12) reduce to a system of equi-dimensional (or Cauchy–Euler) differential equations. The homogeneous solutions (28) then take the form

$$[U_1^h(r), U_2^h(r), U_3^h(r)] = \sum_{i=1}^{6} e_i r^{p_i} [a_{i0}, b_{i0}, c_{i0}]. \tag{31}$$

NUMERICAL EXAMPLES

The elasticity solution derived above has been used to calculate the response of graphite/epoxy doubly-curved panels having dimensions $h = 0.05R_2$ and $\Theta_2 = \Theta_3 = 0.25$. Three-layer regular cross-ply laminates with fiber arrangement [0°/90°/0°] are considered. The thermoelastic properties of an orthotropic lamina with fiber reinforcement in the 2-direction are taken to be:

$$E_1 = 10.3E_0, \quad E_2 = 181.1E_0, \quad E_3 = 10.3E_0,$$
$$G_{12} = 7.17E_0, \quad G_{13} = 2.39E_0, \quad G_{23} = 7.17E_0,$$
$$\nu_{21} = 0.28, \quad \nu_{23} = 0.28, \quad \nu_{13} = 0.28,$$
$$\alpha_1 = 22.5\alpha_0, \quad \alpha_2 = 0.02\alpha_0, \quad \alpha_3 = 22.5\alpha_0,$$

in which standard notation has been employed, and where E_0 and α_0 are reference values.

First, consider the response of a saddle-shape panel for which $R_3 = -2R_2$, subject to the surface tractions described by eqns (9), with $q_{1i} = q_{1o} = q/2$, $q_{2i} = q_{2o} = q_{3i} = q_{3o} = 0$ and $m = n = 1$. Table 1 illustrates the convergence rate of the power series in eqns (20) for this example. It is seen that at least 80 terms must be retained in (20) in order to obtain a relatively accurate value of the center deflection $u_1(1, \Theta_2, \Theta_3)$. The center deflection (u_1)

Table 1. Convergence rate of power series for elasticity solution.
$(h = 0.05R_2,\ \Theta_2 = \Theta_3 = 0.25,\ R_3 = -2R_2)$

No. of terms	$\dfrac{u_1(1, \Theta_2, \Theta_3)}{(q/E_0)}$
60	0.13872
70	−0.22832
80	0.50417
90	0.50348
100	0.50348

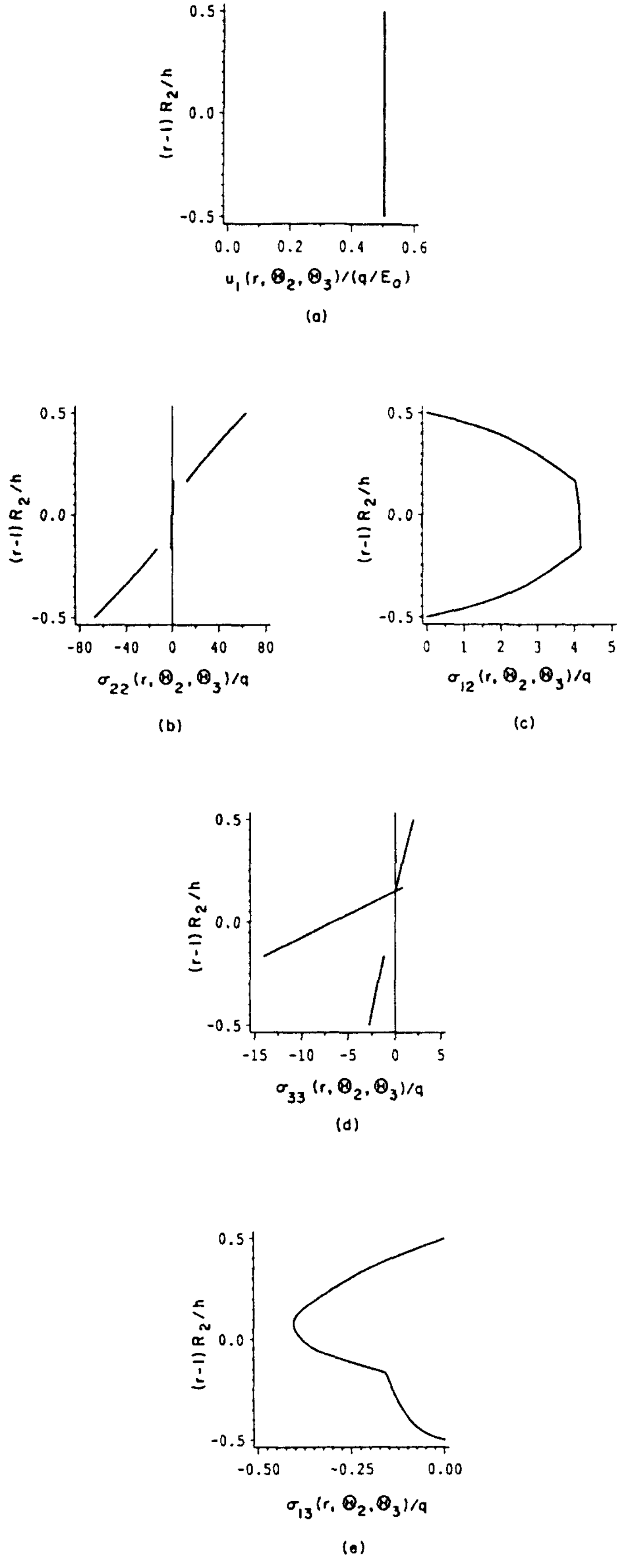

Fig. 2. Reponse of a three-layer doubly-curved laminate with $h = 0.05R_2$, $\Theta_2 = \Theta_3 = 0.25$, $R_1 = -2R_2$.

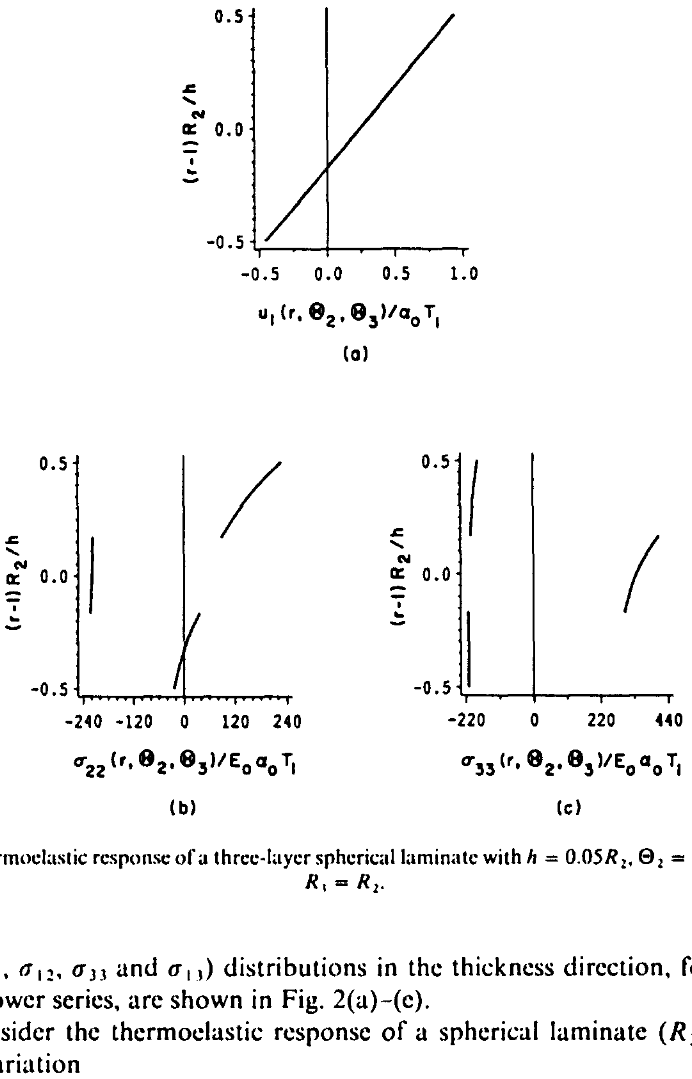

Fig. 3. Thermoelastic response of a three-layer spherical laminate with $h = 0.05R_2$, $\Theta_2 = \Theta_3 = 0.25$, $R_1 = R_2$.

and stress (σ_{22}, σ_{12}, σ_{33} and σ_{13}) distributions in the thickness direction, found using 90 terms in the power series, are shown in Fig. 2(a)–(c).

Next, consider the thermoelastic response of a spherical laminate ($R_3 = R_2$) to the temperature variation

$$T(R, \theta_2, \theta_3) = T_1 \sin(\pi\theta_2/2\Theta_2)\sin(\pi\theta_3/2\Theta_3).$$

The calculated center deflection and stress (σ_{22}, σ_{33}) distributions through the laminate thickness are shown in Fig. 3(a)–(c), respectively.

CONCLUDING REMARKS

The power series method has been used to construct an exact three-dimensional thermoelasticity solution for a doubly-curved cross-ply panel. Solutions such as this have value as benchmarks for verifying mathematical procedures leading to approximate solutions, and are useful for assessing the validity of laminated shell theories. A comparison of results obtained using a recently developed shear–deformation shell theory, with those based upon the present thermoelasticity formulation, will be reported in the near future.

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APPENDIX: SOLUTION TO THE CHARACTERISTIC EQUATION OF EQNS (27)

Let $x = p(p+1)$ and rewrite eqns (27) in the following form:

$$\begin{bmatrix} C_{11}x+k_{11} & -\alpha(C_{12}+C_{66})p+k_{12} & -\beta(C_{13}+C_{55})p+k_{13} \\ \alpha(C_{12}+C_{66})p+k_{21} & C_{66}x+k_{22} & k_{23} \\ \beta(C_{13}+C_{55})p+k_{31} & k_{23} & C_{55}x+k_{33} \end{bmatrix} \begin{Bmatrix} a_0 \\ b_0 \\ c_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

Setting the determinant of the above set of equations to zero gives:

$$e_3x^3+e_2x^2+e_1x+e_0 = 0, \tag{A1}$$

in which

$$e_3 = C_{11}C_{55}C_{66},$$

$$e_2 = C_{11}C_{55}k_{22}+C_{11}C_{66}k_{33}+C_{55}C_{66}k_{11}+\alpha^2C_{55}(C_{12}+C_{66})+\beta^2C_{66}(C_{13}+C_{55})^2,$$

$$e_1 = C_{11}k_{22}k_{33}+C_{55}k_{11}k_{22}+C_{66}k_{11}k_{33}-C_{11}k_{23}^2-C_{55}k_{12}k_{21}-C_{66}k_{13}k_{31}-2\alpha\beta(C_{12}+C_{66})(C_{13}+C_{55})k_{23}$$
$$+\alpha^2(C_{12}+C_{66})^2k_{33}+\beta^2(C_{13}+C_{55})^2k_{22},$$

$$e_0 = k_{11}k_{22}k_{33}+k_{23}k_{12}k_{31}+k_{23}k_{13}k_{21}-k_{11}k_{23}^2-k_{12}k_{21}k_{33}-k_{22}k_{13}k_{31}.$$

The standard solution of eqn (A1) is available (Shelby, 1968). The root p can be obtained by solving the quadratic equation $p(p+1) = x$.